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The Euler–Lagrange theory for Schur's Algorithm: Algebraic exposed points

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To Barry Simon on the occasion of the 60th Birthday

Abstract

In this paper the ideas of Algebraic Number Theory are applied to the Theory of Orthogonal polynomials for *algebraic* measures. The transferring tool are Wall continued fractions. It is shown that any set of closed arcs on the circle supports a quadratic measure and that any algebraic measure is either a Szegö measure or a measure supported by a proper subset of the unit circle consisting of a finite number of closed arcs. Singular parts of algebraic measures are finite sums of point masses.

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1. Introduction

1. Algebraic measures. A probability measure σ with compact support $\operatorname{supp}(\sigma)$ on a rectifiable curve Γ in the complex plane $\mathbb{C}$ is called *algebraic* if its Cauchy transform

$$C_\sigma(z) \stackrel{\text{def}}{=} \int \frac{d\sigma(\zeta)}{\zeta - z} \tag{1}$$

is a branch of an algebraic function on some open subset of each connected component of $\mathbb{C}\backslash\operatorname{supp}(\sigma)$. The reason why probability measures are supposed to have compact support is that

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in case of algebraic measures the inclusion $\infty \in \operatorname{supp}(\sigma)$ implies that not all polynomials belong to $L^2(d\sigma)$.

An example of an algebraic measure is given by Chebyshev's weight:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{z-t}\,\frac{dt}{\sqrt{1-t^2}}=\frac{1}{\sqrt{z^2-1}}. \tag{2}$$

Here $\Gamma = \mathbb{R}$ is the real axis and $\operatorname{supp}(\sigma) = [-1, 1]$.

The importance of algebraic measures originates in the fact that these measures are determined by a finite number of complex parameters. For instance, Chebyshev's measure is determined by the coefficients of the polynomials in the irreducible algebraic equation satisfied by its Cauchy transform:

$$(z^2-1)X^2-1=0. \tag{3}$$

Another example of algebraic measures is given by Akhiezer's weights [2] (see also [20] for a motivation for introducing such measures)

$$w_{\alpha,\beta}(x)=\frac{1}{\pi}\sqrt{\frac{x-\alpha}{(1-x^2)(x-\beta)}},\quad x\in[-1,\alpha]\cup[\beta,1],\ \ -1<\alpha<\beta<1. \tag{4}$$

Then Cauchy's formula shows that

$$\int_{-1}^{1}\frac{w_{\alpha,\beta}(x)\,dx}{z-x}=\sqrt{\frac{z-\alpha}{(z^2-1)(z-\beta)}} \tag{5}$$

satisfies the irreducible quadratic equation

$$(z^2-1)(z-\beta)X^2-(z-\alpha)=0. \tag{6}$$

If σ is an algebraic measure on $\mathbb{R}$, then $(C_\sigma)_\# \stackrel{\text{def}}{=} \overline{C_\sigma(\overline{z})} = C_\sigma(z)$ which implies that the coefficients $\{a_0, a_1, \ldots, a_n\}$ of the irreducible equation for C_σ:

$$a_0X^n+a_1X^{n-1}+\cdots+a_n=0 \tag{7}$$

are polynomials with real coefficients.

Similarly, for σ supported on $\mathbb{T}$, the identity

$$C_\sigma(z)_* \stackrel{\text{def}}{=} \overline{C_\sigma\left(\frac{1}{\overline{z}}\right)}=-z-z^2C_\sigma(z) \tag{8}$$

shows that C_σ is an algebraic function in both domains $\mathbb{D}$ and $\hat{\mathbb{C}}\setminus\operatorname{Clos}\mathbb{D}$ as soon as it is algebraic in one of them.

Let a probability measure μ on $[-1, 1]$ be an image of a symmetric (with respect to $\mathbb{R}$) probability measure σ on $\mathbb{T}$ under the orthogonal projection $\zeta \to (\zeta + 1/\zeta)/2$. Then a well-known formula [14, §30, (30.4)]

$$\int_{\mathbb{T}}\frac{\zeta+z}{\zeta-z}\,d\sigma(\zeta)=\frac{1-z^2}{2z}\int_{-1}^{1}\frac{d\mu(x)}{\frac{1}{2}(z+1/z)-x} \tag{9}$$

reduces the study of algebraic measures with compact support on $\mathbb{R}$ to algebraic measures on $\mathbb{T}$.

2. A number-theoretic motivation. There is a correspondence between problems of Number Theory and Orthogonal Polynomials. This correspondence is realized by continued fractions (see [20] for details). Regular continued fractions are expressions of the form

$$\xi = b_0 + \mathop{\mathbf{K}}_{k=1}^{\infty}\left(\frac{1}{b_k}\right) = b_0 + \frac{1}{b_1} {}_{+\cdots+} \frac{1}{b_n + \xi_n}, \tag{10}$$

where b_n are positive integers. The process in (10) stops in a finite number of steps ($\xi_n = 0$) if and only if $\xi \in \mathbb{Q}$. If we put $\xi_n = 0$ in (10), then the finite continued fraction obtained represents a rational number P_n/Q_n in lowest terms. The fraction P_n/Q_n is called a convergent for (10). Any rational number is a convergent for some continued fraction.

In 1685 Wallis [34] proved that common fractions with denominators $2^p 5^q$ can be transformed into finite decimal fractions. He also showed that the length of the period of the decimal fraction corresponding to a common fraction m/n cannot exceed $n-1$. By stating similar question in respect to continued fractions Euler observed that periodic regular continued fractions represent quadratic irrationalities.

Recall that a regular continued fraction is called *periodic* if there exist $h \in \mathbb{Z}$, $h \geqslant 0$ and $d \in \mathbb{N}$ such that $b_{j+d} = b_j$ for $j = h, h+1, \dots$. If $h = 0$, then a periodic continued fraction is called *purely periodic*. A quadratic irrationality ξ (over $\mathbb{Q}$) is called *reduced* if $\xi > 1$ and the algebraic conjugate irrationality ξ' belongs to the open interval $(-1, 0)$.

Theorem 1.1 (*Euler*). *The value of any regular periodic continued fraction is a quadratic irrationality.*

Theorem 1.2 (*Lagrange*). *The regular continued fraction of a quadratic irrationality is periodic.*

Theorem 1.3 (*Galois [11]*). *A regular continued fraction is purely periodic if and only if its value is a reduced quadratic irrationality.*

See the proofs in [22]. The last theorem was the first published result of Galois, who seemingly attempted to apply continued fractions for the solution of the Basic Theorem of Algebra on algebraic equations.

3. Function fields. It looks attractive to transfer the theory of regular periodic continued fractions developed by Euler and Lagrange for quadratic real irrationalities to the case of function fields. The first contribution here is due to Abel (see [1,20]). Abel considered the function field $\mathbb{C}\,([1/z])$ of formal Laurent series $f(z)$ at $z = \infty$

$$f(z) = \sum_{k \in \mathbb{Z}} \frac{c_k}{z^k}, \tag{11}$$

which has a finite number of non-zero terms with $k < 0$. Following Abel, we put

$$[\![f]\!] = \sum_{k \leqslant 0} \frac{c_k}{z^k}, \quad \mathrm{Frac}(f) = \sum_{k > 0} \frac{c_k}{z^k}.$$

Similar to the case of real numbers $[\![f]\!]$ is called the *integer* part and $\mathrm{Frac}(f)$ the *fractional* part of f. The field $\mathbb{C}\,([1/z])$ is equipped with a non-archimedean norm

$$\|f\| = \exp(\deg(f)), \quad \deg(f) = -\inf\{k \in \mathbb{Z} : c_k \neq 0\}. \tag{12}$$

For a polynomial f ($f \in \mathbb{C}[z]$) $\deg f$ in (12) is the degree of f. We put $\deg(0) = -\infty$.

As in the case of regular continued fractions we put $f_0 = f$ and define $f_n = 1/\mathrm{Frac}(f_{n-1})$ for $n = 1, 2, \ldots$. Then

$$f = [\![f_0]\!] + \frac{1}{1/\mathrm{Frac}(f_0)} = [\![f_0]\!] + \frac{1}{[\![f_1]\!]} + \frac{1}{[\![f_2]\!]} + \cdots + \frac{1}{[\![f_n]\!] + \mathrm{Frac}(f_n)}. \tag{13}$$

It is clear that $b_k = [\![f_k]\!] \in \mathbb{C}[z]$. A finite or infinite continued fraction obtained this way is called a P-fraction (a *polynomial* fraction). Polynomials b_k are the integer elements of the field $\mathbb{C}(z)$ of rational functions.

Euler and Lagrange studied continued fractions of real quadratic irrationalities. Similar one can study continued fractions in $\mathbb{C}([1/z])$ corresponding to quadratic irrationalities. For instance, for Chebyshev's weight:

$$\frac{1}{\pi}\int_{-1}^{1} \frac{1}{z-t}\frac{dt}{\sqrt{1-t^2}} = \frac{1}{\sqrt{z^2-1}}$$
$$= \frac{1}{z} - \frac{1}{2z} - \frac{1}{2z} - \frac{1}{2z} - \cdots = \frac{1}{z} + \frac{1}{-2z} + \frac{1}{2z} + \frac{1}{-2z} + \frac{1}{2z} + \cdots. \tag{14}$$

One may think that quadratic irrationalities in $\mathbb{C}(z, \sqrt{\mathcal{D}})$, where $\mathcal{D}$ is a separable polynomial, correspond (as in the case of real line) to periodic P-fractions. This conjecture is well supported by (2). However, this is not the case even for $\mathcal{D} = z^2 - 1$. To see this consider Laguerre's continued fraction

$$\left(\frac{z+1}{z-1}\right)^{\alpha} = 1 + \frac{2\alpha}{z-\alpha} + \frac{\alpha^2-1}{3z} + \frac{\alpha^2-4}{5z} + \frac{\alpha^2-9}{7z} + \cdots, \tag{15}$$

which can be turned into a P-fraction with the equivalence transform, see [27, §28, (10), 17]. Assuming that $-1 < \alpha < 1$, we obtain by Cauchy's formula

$$\frac{\sin \pi\alpha}{2\pi\alpha}\int_{-1}^{1} \frac{1}{z-x}\left(\frac{1+x}{1-x}\right)^{\alpha} dx = \frac{1}{2\alpha}\left\{\left(\frac{z+1}{z-1}\right)^{\alpha} - 1\right\}, \tag{16}$$

which by (15) implies that

$$\frac{\sin \pi\alpha}{2\pi\alpha}\int_{-1}^{1} \frac{1}{z-x}\left(\frac{1+x}{1-x}\right)^{\alpha} dx = \frac{1}{z-\alpha} - \frac{1}{3z/(1-\alpha^2)} -$$
$$\frac{1}{(1-\alpha^2)5z/(4-\alpha^2)} - \frac{1}{(9-\alpha^2)(4-\alpha^2)7z/(1-\alpha^2)} - \cdots. \tag{17}$$

For $\alpha \neq 0$ the measure $(1+x)^{\alpha}(1-x)^{-\alpha}\,dx$ on $[-1, 1]$ corresponds to the Jacobi polynomials $P_n^{(\alpha,-\alpha)}(x)$. If $\alpha = 1/2$, then (17) shows that the P-fraction corresponding to the Jacobi polynomials $\{P_n^{(1/2,-1/2)}\}_{n\geqslant 0}$ is not periodic. Notice that by (16) the Cauchy integral of the probability measure related with $\{P_n^{(1/2,-1/2)}\}_{n\geqslant 0}$ belongs to $\mathbb{C}(z, \sqrt{z^2-1})$. Hence in contrast to the case of number fields there exists a quadratic field $\mathbb{C}(z, \sqrt{\mathcal{D}})$ with $\sqrt{\mathcal{D}}$ represented by periodic continued P-fractions, which, however, contains an irrationality corresponding to non-periodic P-fractions. Moreover, this quadratic irrationality can be represented as the Cauchy integral of a positive Borel measure. See other examples in [6,7].

4. Schur's Algorithm. By (9) the study of algebraic measures supported on $\mathbb{R}$ is reduced to the study of algebraic measures on $\mathbb{T}$. In case of real line the number-theoretic analogy is guided by P-fractions. In case of the unit circle the guiding tool is Schur's Algorithm or Wall continued fractions.

The Herglotz formula

$$\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\, d\sigma(\zeta) = \frac{1+zf(z)}{1-zf(z)}, \quad z \in \mathbb{D}, \tag{18}$$

determines a homeomorphism $\mathcal{H} : \sigma \to f = \mathcal{H}(\sigma)$ of the convex set of all probability measures $\mathcal{P}(\mathbb{T})$ on $\mathbb{T}$ equipped with the $*$-weak topology onto the unit ball $\mathcal{B}$ of the Hardy algebra H^∞.

For instance for $d\sigma = (1+\cos\theta)\, d\theta/2\pi$ and $|z| < 1$

$$1 + z = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z}(1+\cos\theta)\, d\theta \tag{19}$$

implies that $f(z) = \mathcal{H}(\sigma)(z) = 1/(2+z)$.

Function f in (18) is called the *Schur function* of σ. By the definition of an algebraic measure (8) and (18) imply that σ is algebraic if and only if $f = \mathcal{H}(\sigma)$ is algebraic in $\mathbb{D}$. Thus algebraic measures on $\mathbb{T}$ can be classified with algebraic Schur functions.

Applying Schwarz' lemma to $f = f_0 \in \mathcal{B}$, we obtain a sequence $\{f_n\}_{n\geqslant 0}$ of functions in $\mathcal{B}$ and a sequence $\{a_n\}_{n\geqslant 0}$ of complex numbers in $\mathbb{D}$ such that

$$f_0 = \frac{zf_1(z)+a_0}{1+\bar{a}_0 z f_1(z)}; \ldots ;\, f_n = \frac{zf_{n+1}(z)+a_n}{1+\bar{a}_n z f_{n+1}(z)}; \ldots \quad . \tag{20}$$

Schur's algorithm (20) terminates at the nth step if $|a_n| = 1$ and runs up to infinity if $|a_n| < 1$, $n = 0, 1, \ldots$. It is clear that (20) terminates if and only if f is a rational function in $\mathcal{B}$ satisfying $|f| = 1$ on $\mathbb{T}$. Such functions are called *finite Blaschke products* [12].

The complex numbers a_n are called the *Schur parameters* of f. The functions f_n are called the *Schur functions* (of order n) associated with f.

Applying to (20) elementary algebra, one can move f_{n+1} inside the fraction as far as possible:

$$f_n(z) = \frac{zf_{n+1}(z)+a_n}{1+\overline{a_n} z f_{n+1}(z)} = a_n + \frac{(1-|a_n|^2)z}{\overline{a_n}z + 1/f_{n+1}(z)}.$$

Iterations lead to the Wall continued fraction [33]

$$f(z) = a_0 + \frac{(1-|a_0|^2)z}{\overline{a_0}z} \; \underset{+\, a_1}{\frac{1}{}} \; \underset{+}{} \; \frac{(1-|a_1|^2)z}{\overline{a_1}z} \; \underset{+\cdots}{} \;. \tag{21}$$

Following the analogy with Number Theory, we may pose a problem of a classification of quadratic or more generally algebraic Schur functions in terms of the parameters $\{a_n\}_{n\geqslant 0}$.

If we put $b_k = a$ in (10), then we obtain a quadratic irrationality

$$\xi = \frac{a+\sqrt{a^2+4}}{2},$$

which cannot be rational since the continued fraction (10) is infinite. An especially important example is obtained when $a = 1$. It is the so-called Golden Ratio. Similarly, let $a_k = a$ in (21). Then $f_1 = f$ and (20) implies that f satisfies a quadratic equation

$$\bar{a}zX^2 + (1-z)X - a = 0. \tag{22}$$

The discriminant $\mathcal{D}_a$ of (22) is

$$\mathcal{D}_a = (z-1)^2 + 4|a|^2 z = z\{4|a|^2 - |1-z|^2\}, \quad z \in \mathbb{T}. \tag{23}$$

It follows that

$$\mathcal{T}_a = \frac{\mathcal{D}_a}{z} = 4|a|^2 - |1-z|^2$$

is a real trigonometric polynomial on $\mathbb{T}$. The roots of $\mathcal{D}_a$ are symmetric with respect to the real axis and lie on $\mathbb{T}$:

$$z_{\pm} = 1 - 2|a|^2 \pm 2|a|i\sqrt{1-|a|^2}.$$

The arc $\Delta_\alpha = \{\exp(i\theta) : \alpha \leqslant \theta \leqslant 2\pi - \alpha\}$, where $\sin(\alpha/2) = |a|$, $0 < \alpha < \pi$, connects z_+ with z_- counterclockwise. It follows that

$$\mathcal{T}_a(e^{i\theta}) = 4\left(|a|^2 - \sin^2\frac{\theta}{2}\right) = 2(\cos\theta - \cos\alpha)$$

is negative on Δ_α and is positive on the open complementary arc centered at $z = 1$. Then

$$f_a(z) = \frac{(z-1) + \sqrt{\mathcal{D}_a}}{2az} = \frac{\sqrt{z} - 1/\sqrt{z} + \sqrt{\mathcal{T}_a}}{2a\sqrt{z}} = \frac{2i\ \sin\theta/2 + \sqrt{\mathcal{T}_a}}{2ae^{i\theta/2}},$$

where the branch of the square root is taken to satisfy $\sqrt{1} = 1$, must be unimodular on $\mathbb{T} \setminus \Delta_\alpha$.

Let us consider another example. Already Schur [28] computed the parameters a_n and the Schur functions for $f(z) = 1/(2+z)$. Schur obtained by induction the following formulas:

$$f_n(z) = \frac{1}{(n+1)z + (n+2)}, \quad a_n = \frac{1}{n+2}. \tag{24}$$

It is clear that for $f(z) = 1/(2+z)$ the Wall continued fraction (21) is neither finite (notice that $f \in \mathbb{C}(z)$), nor is periodic. So it looks like that a direct generalization of Theorem 1.2 does not hold. However, the first example with $a_k = a$ fits the theory quite well. The key explanation here is that in the first example the Schur function f is unimodular on an arc of $\mathbb{T}$, whereas in the second it is unimodular only at $z = -1$. This observation leads us to the geometry of the unit ball $\mathcal{B}$ in the Hardy algebra H^∞.

5. The geometry of $\mathcal{B}$ and orthogonal polynomials. Orthogonal polynomials $\varphi_n(z) = k_n z^n + \cdots + \varphi_n(0)$, $k_n > 0$ in $L^2(d\sigma)$ are defined as the outcome of the Gram–Schmidt algorithm applied to the sequence $\{z^n\}_{n\geqslant 0}$ of monomials in $L^2(d\sigma)$. Orthogonal polynomials $\varphi_n(z)$ are uniquely determined by their *Verblunsky parameters*

$$v_n = -\frac{\overline{\varphi_{n+1}(0)}}{k_{n+1}}, \quad n = 0, 1, \ldots . \tag{25}$$

By Geronimus' theorem [13] $a_n = v_n$ for every n. See [18,29,30] for details.

It is known that to a great extent (see [18]) general properties of orthogonal polynomials are determined by the geometry of the convex set $\mathcal{B}$ near the Schur function f of σ. For instance, it was observed by Boyd [4] that σ is a *Szegö measure*, i.e.

$$-\infty < \int_{\mathbb{T}} \log\sigma' dm, \quad \sigma' = \frac{d\sigma}{dm}, \tag{26}$$

if and only if the Schur function $f = \mathcal{H}(\sigma)$ is not an extreme point of $\mathcal{B}$.

Recall that a point x of a convex set V is called *extreme* if x cannot be included in an open interval $(x_1, x_2) \stackrel{\text{def}}{=} \{tx_1 + (1-t)x_2 : 0 < t < 1\}$ with $x_1, x_2 \in V$.

Theorem 1.4 (*K. de Leeuw and W. Rudin [23]*). *A function f is an extremal point of $\mathcal{B}$ if and only if*

$$\int_{\mathbb{T}} \log(1-|f|^2)\,dm = -\infty. \tag{27}$$

Applying Fatou's theorem on non-tangential limits [12, Ch. I, §5], to the real parts of (18), we obtain that

$$\frac{d\sigma}{dm} = \frac{1-|f|^2}{|1-\zeta f(\zeta)|^2}, \quad \zeta \in \mathbb{T}, \tag{28}$$

which shows that

$$\int_{\mathbb{T}} \log \sigma'\,dm = \int_{\mathbb{T}} \log(1-|f|^2)\,dm \tag{29}$$

since $1-\zeta f(\zeta)$ is an outer function. Hence Szegös measures are in one-to-one correspondence with non-extreme points of $\mathcal{B}$.

Similar arguments show that $\sigma' > 0$ a.e. on $\mathbb{T}$, i.e. σ is an Erdös measure if and only if $|f| < 1$ a.e. on $\mathbb{T}$. Schur functions of Erdös measures can also be described in terms of the Banach space geometry of $\mathcal{B}$.

A point x in the unit ball of a Banach space X is called an *exposed point* of ball(X) if there is x^* in the conjugate space X^* such that $\|x^*\| = x^*(x) = 1$ but $|x^*(y)| < 1$ for all $y \in$ ball(X), $y \neq x$.

Exposed points of $\mathcal{B}$ were described by Amar and Lederer [3] who applied the approach developed by Fisher [10]. By Fatou's theorem [12] every element f, $f \in H^\infty$, can be identified with its radial limits on $\mathbb{T}$. Therefore, every $f \in H^\infty$ determines the set

$$\mathcal{U} = \{t \in \mathbb{T} : |f(t)| = 1\} \tag{30}$$

up to a subset of m-measure zero.

Theorem 1.5 (*Amar and Lederer [3]*). *A function f is an exposed point of $\mathcal{B}$ if and only if $m(\mathcal{U}(f)) > 0$.*

We refer the reader to an interesting survey [15] for the proof of this theorem as well as for a discussion of a closely related topics.

By Theorem 1.5 a probability measure σ is an Erdös measure if and only if $f = \mathcal{H}(\sigma)$ is a non-exposed point of $\mathcal{B}$. We denote by Exp$(\mathcal{B})$ the set of all exposed points of $\mathcal{B}$. If $\sigma \in \mathcal{P}$, then

$$\mathcal{L}(\sigma) \stackrel{\text{def}}{=} \{t \in \mathbb{T} : \sigma'(t) > 0\} \tag{31}$$

is the Lebesgue support of σ. The sets $\mathcal{U}(f)$ and $\mathcal{L}(\sigma)$ do not intersect and cover $\mathbb{T}$ by modulus a subset of m-measure zero.

6. Main results. Since algebraic measures on $\mathbb{T}$ as well as their Schur functions depend on a finite number of parameters (the coefficients of the corresponding algebraic equation), it is natural to expect that they may ignore such delicate difference as that between Szegö and Erdös classes, or equivalently between non-extreme and non-exposed points of $\mathcal{B}$. On the other hand this very property of algebraic measures may lead to the existence of simple invariants which may be useful for their classifications.

The Schur function f of any algebraic measure σ can be uniquely extended to a continuous function on the closure $\mathrm{Clos}(\mathbb{D})$ of $\mathbb{D}$ (see Lemma 2.4). It follows that $\mathcal{U}(f)$ is a closed subset of $\mathbb{T}$ if σ is an algebraic measure. We show that $\mathcal{U}(f)$ is the required invariant for the classification of algebraic measures.

Given any subset E of a metric topological space X we denote by E' the *derived set* of E, i.e. the set of all limit points of E in X, and by $\overset{\circ}{E}$ the set of all interior points of E.

Theorem 3.1. *Let f be an algebraic point of $\mathcal{B}$. Then either f is the Schur function of a Szegö measure or f is an exposed point of $\mathcal{B}$. The first case occurs if and only if $\mathrm{Card}\,\mathcal{U}(f) < +\infty$, whereas the second occurs if and only if $\mathcal{U}(f)'$ is a union of a finite number of non-empty non-intersecting closed arcs on $\mathbb{T}$ with finite number of points on $\mathbb{T}$.*

Elementary calculations

$$1 - |f_n|^2 = \frac{(1-|a_n|^2)(1-|f_{n+1}|^2)}{|1+\bar{a}_n z f_{n+1}|^2}, \quad z \in \mathbb{T}, \tag{32}$$

imply $\mathcal{U}(f) = \mathcal{U}(f_n)$, $n = 0, 1, 2, \ldots$. Hence by Theorem 1.5 the set $\mathrm{Exp}(\mathcal{B})$ is invariant under Schur's transforms. Similarly, the set of Schur functions of Szegö measures is invariant under Schur's transforms by Theorem 1.4. Theorem 3.1 shows that algebraic measures are invariant under Schur's transforms. A technique for algebraic measures in Szegö's case is presented in a recent paper [5]. The present paper mostly is directed to the study of exposed algebraic irrationalities.

Theorem 5.7. *A quadratic field $\mathbb{C}(z, \sqrt{\mathcal{D}})$ contains an element corresponding to an exposed point of $\mathcal{B}$ if and only if $\mathcal{D}$ is a separable polynomial of even degree with roots on $\mathbb{T}$.*

Corollary 5.10. *For any family of disjoint closed arcs $\{\gamma_j\}_{j=1}^{L}$ there is an exposed quadratic irrationality f in $\mathcal{B}$ such that*

$$\mathcal{U}(f) = \bigcup_{j=1}^{L} \gamma_j.$$

We denote by $\mathrm{Exp}_a(\mathcal{B})$ the set of algebraic exposed points in $\mathcal{B}$. To study $\mathrm{Exp}_a(\mathcal{B})$ we need some preliminaries on algebraic functions.

2. Algebraic preliminaries

7. $*$-Reversed polynomials. Let $\mathbb{C}[z]$ be the set of all polynomials p in z. We denote by $\deg p$ the degree of p. Every non-zero polynomial $p \in \mathbb{C}[z]$ determines the conjugate polynomial

$$p^*(z) = z^{\deg p}\, \overline{p\left(\frac{1}{\bar{z}}\right)}, \quad z \in \mathbb{C}. \tag{33}$$

Definition 2.1. A polynomial p is called $*$-invariant if there exists a unimodular constant $\lambda(p)$ such that

$$p^* = \lambda(p)p. \tag{34}$$

We denote by $^*\mathbb{C}[z]$ the set of all $*$-invariant polynomials. It follows from the definition that $p(0) \neq 0$ for every $p \in {}^*\mathbb{C}[z]$ and that $^*\mathbb{C}[z]$ is a multiplicative semigroup. The mapping $p \to \lambda(p)$ is a character of $^*\mathbb{C}[z]$, i.e. a multiplicative homomorphism of $^*\mathbb{C}[z]$ to the multiplicative group $\mathbb{T}$. It is easy to see that

$$\lambda(p) \quad = \quad \begin{cases} \bar{t}^2 & \text{if } p \equiv t, \quad t \in \mathbb{T}, \\ -\bar{t} & \text{if } p = z - t, \quad t \in \mathbb{T}, \\ \bar{t}^2 & \text{if } p = (z - rt)(z - t/r), \quad t \in \mathbb{T}, \ \ r > 0\,. \end{cases} \tag{35}$$

It follows from (35) that any polynomial with roots on $\mathbb{T}$ is an element of $^*\mathbb{C}[z]$.

For any $p \in \mathbb{C}[z]$ and $\zeta \in \mathbb{T}$ we denote by $p_\zeta(z) \stackrel{\text{def}}{=} p(\bar{\zeta}z)$ the "rotation" of p by the unit vector ζ. The mapping $p \to p_\zeta$ is an endomorphism of $^*\mathbb{C}[z]$. Since

$$p_\zeta^* = z^{\deg p}\,\bar{p}_\zeta = \zeta^{\deg p}\,p^*(\bar{\zeta}z) = \lambda(p)\zeta^{\deg p}\,p_\zeta,$$

we obtain the following formula

$$\lambda(p_\zeta) = \lambda(p)\zeta^{\deg p}. \tag{36}$$

In what follows we sometime consider polynomials p as elements of the linear space

$$\mathbb{C}_n[z] \stackrel{\text{def}}{=} \{\, p \in \mathbb{C}[z] : \ \deg p \leqslant n \,\}.$$

In this case we define the operation

$$(p)_n^* = z^{n-\deg p}\,p^*. \tag{37}$$

When there is no danger of misleading we just write $p^* = (p)_n^*$, especially if the degree n is reflected in the index.

8. Algebraic functions. Historically algebraic functions appeared as functions of complex variable z satisfying irreducible algebraic equations

$$p(z, X) \stackrel{\text{def}}{=} \alpha_0(z)X^g + \alpha_1(z)X^{g-1} + \cdots + \alpha_g(z) = 0 \tag{38}$$

with polynomial coefficients $\alpha_j \in \mathbb{C}[z]$, $j = 0, 1, \ldots, g$. The integer g is called the degree of $p(z, X)$ (in X) if $\alpha_0 \not\equiv 0$.

Let $\mathbb{C}[z] \subset \mathfrak{R}$ be an extension of entire rings. Then $p(z, X)$ can be considered as an element of $\mathfrak{R}[X]$. A polynomial p is said to be *irreducible* (over $\mathfrak{R}$) if $p = ab$ in $\mathfrak{R}[X]$ implies that either a or b is invertible in $\mathfrak{R}[X]$ [32, Ch. III, §18].

Suppose that $\mathfrak{R} = \mathbb{C}[z]$. Then the coefficients $\alpha_0, \alpha_1, \ldots, \alpha_g$ cannot have a common divisor if p is irreducible over $\mathbb{C}[z]$. However, these coefficients may have a common divisor if $\mathfrak{R} = \mathbb{C}(z)$ is the quotient field of $\mathbb{C}[z]$.

By Steinitz' theorem [32, Ch. X, §72] there is an algebraically closed extension K of the field $\mathbb{C}(z)$. It follows that in $K[X]$ the polynomial p can be factored as

$$p(z, X) = \alpha_0(X - \theta_1)\cdots(X - \theta_g),$$

where $\theta_j \in K$, $j = 1, \ldots, g$. The smallest subfield of K containing $\mathbb{C}(z)$ and the elements $\theta_1, \ldots, \theta_g$ is called *the splitting field* $\mathbb{C}(z, p)$ of an irreducible polynomial p (over $\mathbb{C}(z)$). Then the elements $\theta_1, \ldots, \theta_g$ in the splitting field $\mathbb{C}(z, p)$ are called algebraic functions associated with algebraic equation (38). Purely algebraic properties of algebraic functions are reflected in the algebraic structures of the finite algebraic extension $\mathbb{C}(z) \subset \mathbb{C}(z, p)$.

The splitting field of (38) can also be constructed with analytic tools. This approach to algebraic functions is based on the theory of resultants. Let q be a polynomial in X of degree e with coefficients $\beta_j \in \mathbb{C}[z]$, $j = 0, 1, \dots, e$. Then the resultant $R(p, q)$ is defined by the following determinant:

$$R(p,q) = \begin{vmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_g & 0 & \dots & \dots & 0 \\ 0 & \alpha_0 & \alpha_1 & \dots & \alpha_g & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & & & \\ 0 & 0 & \dots & \alpha_0 & \alpha_1 & \dots & \dots & \alpha_g \\ \beta_0 & \beta_1 & \dots & \beta_e & 0 & \dots & \dots & 0 \\ 0 & \beta_0 & \beta_1 & \dots & \beta_e & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & & & \\ 0 & 0 & \dots & \beta_0 & \beta_1 & \dots & \dots & \beta_e \end{vmatrix}. \tag{39}$$

It is clear from the definition that $R(p, q) \in \mathbb{C}[z]$. On the other hand

$$R(p,q) = \alpha_0^e \beta_0^g \prod_{i=1}^{g} \prod_{j=1}^{e} (\theta_i - \xi_j), \tag{40}$$

where ξ_j, $j = 1, \dots, e$ are the roots of q in the same algebraic closure of $\mathbb{C}(z)$, which contains $\theta_1, \dots, \theta_g$ [32, Ch. V, §10, Proposition 4]. This implies that the polynomial $R(p, q)$ is identical zero if and only if p and q have a common zero in an algebraic closure of $\mathbb{C}(z)$. In particular, if p is irreducible, then $R(p, q) = 0$ if and only if p divides q.

If we put $z = z_0$ in (38) and (39), then we obtain the polynomials with complex coefficients. It follows from (39) that the resultant of these polynomials in $\mathbb{C}[X]$ is nothing but $R(p, q)(z_0)$. Since $\mathbb{C}$ is algebraically closed, we obtain from the general formula (40) the expression for $R(p, q)(z_0)$ in terms of the product $\alpha_0(z_0)^e \beta_0(z_0)^g$ and the differences of complex roots of $p(z_0, X) = 0$ and $q(z_0, X) = 0$.

Applying the above arguments to the pair p and $q = \dot{p}_X$, where $\dot{p}_X$ is the derivative of p in variable X, we arrive to the following known conclusion [32, §34]: *The resultant $R(p, \dot{p}_X)$ vanishes at a point $a \in \mathbb{C}$ if and only if either $\alpha_0(a) = 0$ or the polynomial $p(a, X)$ with complex coefficients has a multiple root.*

Notice that $R(p, \dot{p}_X)$ cannot vanish identically, since $\dot{p}_X \not\equiv 0$ and p cannot divide $\dot{p}_X$ ($\deg p < \deg \dot{p}_X$). The polynomial $R(p, \dot{p}_X)$ may have only a finite number of zeros. Suppose that $R(p, \dot{p}_X)(a) \neq 0$. Then $\alpha_0(a) \neq 0$ and all roots of the equation $p(a, X) = 0$ are different. By the implicit function theorem there is a small neighborhood $\mathcal{O}$ of a such that (38) has g roots $w_1(z), w_2(z), \dots, w_g(z)$ holomorphic in $\mathcal{O}$ with $w_i(z) \neq w_j(z)$ if $i < j$, $z \in \mathcal{O}$. It follows that (38) has g different roots in the field $\mathcal{M}(\mathcal{O})$ of all meromorphic functions in $\mathcal{O}$. It is clear that $\mathcal{M}(\mathcal{O})$ is an extension of $\mathbb{C}(z)$. By [32, Ch. VIII, §2, Theorem 3] the smallest subfield of $\mathcal{M}(\mathcal{O})$ generated by $\mathbb{C}(z)$ and by $w_1(z), \dots, w_g(z)$ is isomorphic to $\mathbb{C}(z, p)$.

Together with the resultant $R(p, \dot{p}_X)$ one can consider a closely related polynomial called the *discriminant* of p. The expression

$$\mathcal{D}_p = \alpha_0^{2g-2} \prod_{i<j} (w_i - w_j)^2 \tag{41}$$

is a symmetric function of the roots $w_1, \dots, w_g$ and therefore $\mathcal{D}_p \in \mathbb{C}(z)$ by the fundamental theorem for symmetric functions [32, §37]. The polynomial $\mathcal{D}_p$ is called the discriminant of p. It is related to $R(p, \dot{p}_X)$ by the following formula:

$$R(p, \dot{p}_X) = \alpha_0(-1)^{\frac{g(g-1)}{2}} \mathcal{D}_p \tag{42}$$

The third approach to algebraic functions is based on the theory of finitely valued analytic functions on $\mathbb{C}$ [16]. A finitely valued analytic function with a finite number of singular points is called *algebraic* if every singular point of this function is algebraic. The equivalence of the third definition to that given above is established in [16, Ch. IV, §4, Theorem 3].

The fourth, the modern approach, to algebraic functions is based on the theory of Riemann surfaces [9].

The first three approaches depend on the theory of symmetric functions [21, Ch. 1, §3]. In addition to the references already mentioned we refer the reader to an interesting book [25].

9. $*$-transform. The formula

$$\alpha_*(z) = \overline{\alpha\left(\frac{1}{\bar{z}}\right)}, \quad z \in \mathbb{C}, \tag{43}$$

determines a multiplicative semi-linear (i.e. $(\lambda\alpha)_* = \bar{\lambda}\alpha_*$, $\lambda \in \mathbb{C}$) mapping of the field $\mathbb{C}(z)$ onto itself. An application of (43) to (38) extends this mapping to the mapping of the splitting fields $\mathbb{C}(z, p) \to \mathbb{C}(z, p^*)$. It follows that if $w_1(z), w_2(z), \dots, w_g(z)$ are the branches of the algebraic function w corresponding to an irreducible polynomial p, then $(w_1)_*(z), (w_2)_*(z), \dots, (w_g)_*(z)$ are the branches of the algebraic function w_* corresponding to

$$p^*(z, X) = (\alpha_0)^*_L X^g + (\alpha_1)^*_L X^{g-1} + \cdots + (\alpha_g)^*_L = 0, \tag{44}$$

where $L = \max(\deg(\alpha_0), \deg(\alpha_1), \dots, \deg(\alpha_g))$. Since the mapping defined by (43) is multiplicative and semi-linear and since $\alpha = \alpha_{**}$, we obtain that p^* is an irreducible polynomial. We have

$$\mathcal{D}_{p^*}(z) = z^{L(2g-2)} \left(\mathcal{D}_p\right)_* (z), \quad z \in \mathbb{C}, \tag{45}$$

which, by the way, implies that $\deg(\mathcal{D}_p) \leqslant L(2g-2)$ and that the zeros of $\mathcal{D}_p$ and of $\mathcal{D}_{p^*}$ coincide on $\mathbb{T}$. By the uniqueness theorem it is sufficient to prove (45) on $\mathbb{T}$. Taking into account (44), we obtain by (41) that

$$\mathcal{D}_{p^*}(z) = \left(z^L \overline{\alpha_0(z)}\right)^{2g-2} \prod_{i<j} (\bar{w}_i - \bar{w}_j)^2 = z^{L(2g-2)} \overline{\mathcal{D}_p(z)}, \quad z \in \mathbb{T},$$

as stated.

10. Some remarks. We conclude this section with a few remarks on the behavior of the branches $w_1(z), \dots, w_g(z)$ near the zeros of the resultant. To avoid miracle singularities we may assume that the polynomials $\alpha_0, \alpha_1, \dots, \alpha_g$ have no common roots in $\mathbb{C}$, i.e. $\max_{0 \leqslant k \leqslant g} |\alpha_k(z)| > 0$ on $\mathbb{C}$.

Lemma 2.2. *A point $a \in \mathbb{C}$ is a zero of α_0 if and only if there is a branch $w_k(z)$ of the algebraic function associated with* (38), *which is unbounded in a neighborhood of a. A point $a \in \mathbb{C}$ is a zero of α_g if and only if there is a branch w_k assuming arbitrary small values at a neighborhood of a.*

Proof. Suppose that w_k is unbounded near a. Then dividing both sides of (38) by $w_k(z)^g$ and passing to the limit along a subsequence, we obtain that $\alpha_0(a) = 0$.

Suppose now that $\alpha_0(a) = 0$. Then there exists an index k such that $\alpha_k(a) \neq 0$. It follows that the rational function α_k/α_0 is unbounded near a. On the other hand, $(-1)^k\alpha_k/\alpha_0$ is a basic symmetric polynomial of $w_1, \ldots, w_g$. Hence at least one of w_j must be unbounded near a.

The second statement of the lemma follows from the first by the consideration of the "dual" equation with the roots $1/w_1, \ldots, 1/w_g$. □

Definition 2.3. A holomorphic function f in a domain $\mathcal{O}$ is called algebraic if there is an algebraic function w with an irreducible polynomial (38) and a branch w_j, $1 \leqslant j \leqslant g$, of w such that

$$w_j(z) = f(z), \quad z \in \mathcal{O}. \tag{46}$$

We are mainly concerned in this paper with the case $\mathcal{O} = \mathbb{D}$. Recall that the Banach algebra of all continuous functions on $\mathrm{Clos}(\mathbb{D})$ holomorphic on $\mathbb{D}$ is called the disc algebra C_A [12].

Lemma 2.4. *Any algebraic function in $\mathcal{B}$ is an element of the disc algebra.*

Proof. Let f be an algebraic function in $\mathcal{B}$. To prove that f is a restriction of a continuous function on $\mathrm{Clos}(\mathbb{D})$ it is sufficient to show that the branch w_j of the algebraic function (38) defined by (46) is continuous at every singular point on $\mathbb{T}$. Let t be a singular point on $\mathbb{T}$ for w. Since f is uniformly bounded, t cannot be a pole for w_j. If t is an algebraic singular point, then

$$f(z) = w_j(z) = \sum_{n \geqslant s} c_n (z-t)^{n/r}, \quad r \in \mathbb{N}, \tag{47}$$

where the series converges uniformly in $\{z : |z-t| < \varepsilon\} \cap \mathbb{D}$. Since f is bounded, we have $s \geqslant 0$. Observing that the functions $(z-t)^{n/r}$ are in C_A, we obtain the result. □

3. Szegö's alternative for algebraic functions

11. In this section we prove the following theorem.

Theorem 3.1. *Let f be an algebraic point of $\mathcal{B}$. Then either f is the Schur function of a Szegö measure or f is an exposed point of $\mathcal{B}$. The first case occurs if and only if $\mathrm{Card}\,\mathcal{U}(f) < +\infty$, whereas the second occurs if and only if $\mathcal{U}(f)'$ is a union of a finite subset of $\mathbb{T}$ with a finite number of non-empty non-intersecting closed arcs on $\mathbb{T}$.*

Proof. Since both polynomials p and p^* are irreducible, their resultants cannot vanish identically. By (42), (44) and (45) the zero sets of the resultants $R(p, \dot{p}_X)$ and $R(p^*, \dot{p}^*_X)$ on $\mathbb{T}$ coincide. It follows that

$$F(p) \stackrel{\text{def}}{=} \{t \in \mathbb{T} : R(p, \dot{p}_X)(t) = 0\} = \{t \in \mathbb{T} : R(p^*, \dot{p}^*_X)(t) = 0\} \tag{48}$$

is finite. Hence the branches $w_1, \ldots, w_d$ and $w_{1*}, \ldots, w_{d*}$ of the algebraic functions w and w_* are holomorphic on every complementary arc Δ of $F(p)$ in $\mathbb{T}$.

Lemma 3.2. *For every complementary arc* Δ *of* $F(p)$ *defined in* (48) *either* $|f| = 1$ *on* Δ *or Card* $(\mathcal{U}(f) \cap \Delta) < +\infty$ *and*

$$\int_\Delta \log(1 - |f|^2)\, dm > -\infty \tag{49}$$

Proof. Suppose that $|f| \not\equiv 1$ on Δ. Then

$$0 \leqslant 1 - |f|^2 = 1 - w_j w_{j*}$$

is a non-zero algebraic function holomorphic on Δ. Since any algebraic function has a finite number of zeros, there is only a finite number of points t in Δ with $|f(t)| = 1$. Since $1 - w_j w_{j*}$ is holomorphic at t, there are an integer s, $s \geqslant 1$, and $\lambda \in \mathbb{C}$, $\lambda \neq 0$, such that

$$1 - |f(\zeta)|^2 = 1 - w_j(\zeta) w_{j*}(\zeta) = \lambda(\zeta - t)^s (1 + o(1)), \quad \zeta \to t.$$

Next, if t is an end-point of Δ, then either t is a regular point of $1 - w_j w_{j*}$, and then we have the alternative, which has already been considered, or t is an algebraic singular point for $1 - w_j w_{j*}$. Since $w_j \in C_A$, the value $f(t)$ is well-defined. We have either $|f(t)| < 1$, or

$$1 - |f(\zeta)|^2 = \lambda(\zeta - t)^{s/r}(1 + o(1)), \quad \zeta \to t\,, \;\; \zeta \in \Delta,$$

where $\lambda \neq 0$, $s, r \in \mathbb{N}$. It follows that at any point of a finite set $\mathcal{U}(f) \cap \Delta$, as well as at the end points of Δ, we have

$$\log(1 - |f(\zeta)|^2) = s \, \log(\zeta - t) + O(1)\,, \quad \zeta \to t\,,$$

where $s \geqslant 0$. This obviously yields (49). $\square$

By Lemma 3.2 we have two possibilities. Either (49) holds for every complementary arc Δ of $F(p)$ or there are open complementary arcs Δ on which $|f| \equiv 1$. If the first possibility occurs, then Card $\mathcal{U}(f) < +\infty$ by Lemma 3.2 and

$$\int_{\mathbb{T}} \log(1 - |f|^2)\, dm > -\infty,$$

which implies that f is the Schur function of a Szegő measure. If the second possibility occurs, then we have two non-empty classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of complementary arcs Δ. An arc Δ belongs to $\mathcal{C}_1$ if and only if $|f| \equiv 1$ on Δ. An arc Δ belongs to $\mathcal{C}_2$ if Card $(\Delta \cap \mathcal{U}(f)) < +\infty$. It is clear that a non-empty open set

$$\bigcup_{\Delta \in \mathcal{C}_1} \Delta$$

is a dense subset of $\mathcal{U}(f)'$ on $\mathbb{T}$. It follows that the derived set $\mathcal{U}(f)'$ is a finite union of closed arcs.

As to the family $\mathcal{C}_2$, Lemma 3.2 shows that

$$-\infty < \sum_{\Delta \in \mathcal{C}_2} \int_\Delta \log(1 - |f|^2)\, dm = \int_{\mathbb{T} \setminus \mathcal{U}(f)} \log(1 - |f|^2)\, dm.$$

Next, by Fatou's theorem [12, Ch. I, §5, Theorem 5.3]

$$\int_{\mathcal{L}(\sigma)} \log \sigma' \, dm = \int_{\mathbb{T}\setminus\mathcal{U}(f)} \log(1 - |f|^2) \, dm + \int_{\mathcal{U}(f)} \log|1 - zf|^2 \, dm > -\infty. \tag{50}$$

The second integral in the right-hand side of (50) is finite, since $1 - zf$ is an outer function in $\mathbb{D}$ [12, Ch. II, §4]. □

Corollary 3.3. *Let f be the Schur function of an algebraic probability measure σ on $\mathbb{T}$. Then $\mathcal{U}(f)'$ is a union of a finite number of closed arcs with the end points at algebraic singularities of f and*

$$\int_{\mathcal{L}(\sigma)} \log \sigma' \, dm > -\infty.$$

Proof. If $t \in \mathbb{T}$ is a boundary point of $\mathcal{U}(f)'$ on $\mathbb{T}$, then there are two adjacent to t open arcs Δ_0 and Δ_1 such that $|f| < 1$ on Δ_0 and $|f| = 1$ on Δ_1. Let us suppose that t is a regular point of f. The conformal mapping

$$u(z) = t\,\frac{i \pm z}{i \mp z}$$

maps the real line $\mathbb{R}$ onto $\mathbb{T} \setminus \{-t\}$, $u(0) = t$. We can fix the choice of signs by the requirement that a small interval $(0, \delta)$ is mapped in Δ_1. Now we consider an auxiliary function G holomorphic at $z = 0$ and defined by

$$G(z) = -i\,\frac{f(u) - f(t)}{f(u) + f(t)}.$$

Since $|f(u(x))| = 1$ for $x \in [0, \delta]$, we obtain that G is real-valued on $(0, \delta)$, which implies that the Taylor series of G at $z = 0$ has real coefficients. Since G is holomorphic at $z = 0$, $G(x)$ equals the sum of its Taylor series for $\delta_1 < x < 0$. It follows that $G(x)$ is real for such an x. Then $|f(u(x))| = 1$ if $\delta_1 < x < 0$. This, however, contradicts to the condition that $|f| < 1$ on Δ_0.

Since t is not a regular point of f, it is a singular point of the algebraic function f. Since $f \in C_A$, t cannot be a pole of f. It follows that t is an algebraic singular point, as stated.

The equality $E(\sigma) = \mathbb{T} \setminus \mathcal{U}(f)$ follows from the formula (28), which by (50) implies the inequality stated.

We observe that $\mathcal{U}(f) = \mathbb{T}$ if and only if f is a finite Blaschke product. Indeed, if $\mathcal{U}(f) = \mathbb{T}$, then f is a unimodular function in the disc algebra C_A. Then f is a finite Blaschke product [12]. The converse is obvious. Notice, that finite Blaschke products correspond to the terminating case of Schur's algorithm. □

Lemma 3.4. *Let σ be an algebraic measure on $\mathbb{T}$, σ_a the absolutely continuous part of σ, $f = \mathcal{H}(\sigma)$. Then*

$$supp(\sigma_a) = \mathbb{T} \setminus \overset{\circ}{\mathcal{U}}(f), \tag{51}$$

$$d\sigma = d\sigma_a + \sum_{j=1}^{N} p_j \delta_{\zeta_j}, \quad \zeta_j \in \mathbb{T}, \quad p_j > 0. \tag{52}$$

Proof. If σ is a Szegő measure, then by Theorem 3.1 Card $\mathcal{U}(f) < \infty$. Therefore $\mathring{\mathcal{U}}(f) = \emptyset$, which obviously implies (51). If $\sigma \in \mathrm{Exp}(\mathcal{B})$, then by Theorem 3.1 $\mathcal{U}(f)'$ is a finite union of disjoint closed arcs on $\mathbb{T}$. Hence

$$\mathcal{U}(f) = \mathring{\mathcal{U}}(f) \cup F, \quad F \cap \mathring{\mathcal{U}}(f) = \emptyset, \;\; \text{Card}\, F < +\infty,$$

which implies (51) by (28).

Since any algebraic function may have only a finite number of zeros (see Lemma 2.2), the number N in (52) cannot exceed the number of zeros of $zf(z) = 1$ on $\mathbb{T}$. On the other hand, a purely singular part cannot be present in σ, since by (18) C_σ may have only a finite number of singularities. This proves (52). □

4. Exposed algebraic irrationalities

12. The following theorem gives a simple necessary condition in order that an algebraic point of $\mathcal{B}$ be an exposed point. Let us observe that for any irreducible polynomial (38) the coefficients α_0 and α_g are non-zero polynomials in $\mathbb{C}[z]$.

Theorem 4.1. *Let f be an algebraic exposed point in $\mathcal{B}$ and $p(z, X)$ the irreducible polynomial* (38) *associated with the algebraic function w corresponding to f by* (46). *Then*

$$\frac{\alpha_1}{\alpha_0} = \frac{\alpha_{g-1*}}{\alpha_{g*}}; \; \frac{\alpha_2}{\alpha_0} = \frac{\alpha_{g-2*}}{\alpha_{g*}}; \ldots ; \frac{\alpha_g}{\alpha_0} = \frac{\alpha_{0*}}{\alpha_{g*}}. \tag{53}$$

Proof. By Theorem 3.1 there exists a complementary arc Δ of F on $\mathbb{T}$ such that

$$0 = 1 - |f|^2 = 1 - w_j w_{j*}, \quad t \in \Delta. \tag{54}$$

Let G be any simple connected domain with the following properties:

$$\Delta \subset G, \tag{a}$$

$$G \text{ is invariant under } z \to 1/\bar{z}, \tag{b}$$

$$R(p, \dot{p}_X)(z) R(p^*, \dot{p}^*_X)(z) \neq 0, \quad z \in G. \tag{c}$$

Then the branches $w_1, \ldots, w_g, w_{1*}, \ldots, w_{g*}$ are holomorphic in G by the monodromy theorem [8, Ch. III, Theorem 1.2]. Since $w_j w_{j*} = 1$ on Δ by (54), we have

$$w_j(z) = \frac{1}{w_{j*}(z)}, \quad z \in G. \tag{55}$$

Let us observe that $1/w_{j*}$ satisfies in G the algebraic equation

$$(\alpha_g)^*_L X^g + (\alpha_{g-1})^*_L X^{g-1} + \cdots + (\alpha_0)^*_L = 0, \tag{56}$$

which is obviously equivalent to $p^*(z, w_{j*}) = 0$, $z \in G$ and consequently to $p(z, w_j) \equiv 0$.

We consider two monic polynomials over $\mathbb{C}(z)$:

$$p(X) = X^g + \frac{\alpha_1}{\alpha_0} X^{g-1} + \frac{\alpha_2}{\alpha_0} X^{g-2} + \cdots + \frac{\alpha_1}{\alpha_0},$$

$$q(X) = X^g + \frac{(\alpha_{g-1})^*_L}{(\alpha_g)^*_L} X^{g-1} + \frac{(\alpha_{g-1})^*_L}{(\alpha_g)^*_L} X^{g-2} + \cdots + \frac{(\alpha_0)^*_L}{(\alpha_g)^*_L}. \tag{57}$$

The polynomial p is irreducible. By (55) p and q have a common root in the field $K = \mathcal{M}(G) \supset \mathbb{C}(z) = k$. It follows that p divides q and we obtain that $p = q$. Comparing the coefficients in (57), we obtain (53). $\square$

13. With any algebraic function w we associate two rational functions:

$$\begin{aligned} Tr(w) &= w_1 + w_2 + \cdots + w_g = -\frac{\alpha_1}{\alpha_0}, \\ N(w) &= w_1 w_2 \dots w_g = (-1)^g \frac{\alpha_g}{\alpha_0}, \end{aligned} \tag{58}$$

which are called the *trace* $Tr(w)$ and the *norm* $N(w)$ of w.

Corollary 4.2. *Let f be an algebraic exposed point of $\mathcal{B}$ corresponding to an algebraic function w associated with an irreducible polynomial $p(z, X)$. Then*

(a) *the norm $N(w)$ is a unimodular rational function on $\mathbb{T}$;*
(b) *the coefficients $\{\alpha_k\}_{k=0}^{g}$ of p satisfy the following equations on $\mathbb{T}$*

$$\alpha_k = (-1)^g N(w) \frac{\alpha_0}{\bar{\alpha}_0} \bar{\alpha}_{g-k}, \quad k = 0, 1, \dots, g; \tag{59}$$

(c) *the discriminant $\mathcal{D}_p$ satisfies*

$$\mathcal{D}_{p^*}(z) = \left(\frac{\alpha_0^*}{N(w)} \right)^{2g-2} \mathcal{D}_p(z) \tag{60}$$

on the unit circle $\mathbb{T}$.

Proof. (a) Follows from the last identity (53), which implies that $|\alpha_g|^2 = |\alpha_0|^2$ on $\mathbb{T}$.

To prove (59) we observe that by (53) and by the definition of $N(w)$ we have

$$\frac{\alpha_k}{\alpha_0} = \frac{\bar{\alpha}_{g-k}}{\bar{\alpha}_g} = \frac{\alpha_g \bar{\alpha}_{g-k}}{|\alpha_g|^2} = \frac{\alpha_g}{\alpha_0} \frac{\bar{\alpha}_{g-k}}{\bar{\alpha}_0} = (-1)^g N(w) \frac{\bar{\alpha}_{g-k}}{\bar{\alpha}_0},$$

which is equivalent to (59).

Since the polynomials p and q in (57) are equal, there is a permutation τ of the set $\{1, 2, \dots, g\}$ such that

$$\frac{1}{w_{k*}(z)} = w_{\tau(k)}, \quad z \in G, \ \ k = 1, \dots, g.$$

Hence we have in G:

$$\mathcal{D}_{p^*}(z) = (\alpha_0^*)^{2g-2} \prod_{i<j} (w_{i*} - w_{j*})^2 = (\alpha_0^*)^{2g-2} \prod_{i<j} \left(\frac{1}{w_{\tau(i)}} - \frac{1}{w_{\tau(j)}} \right)^2. \tag{61}$$

But

$$\left(\frac{1}{w_{\tau(i)}} - \frac{1}{w_{\tau(j)}} \right)^2 = -\left(\frac{1}{w_{\tau(i)}} - \frac{1}{w_{\tau(j)}} \right) \times \left(\frac{1}{w_{\tau(j)}} - \frac{1}{w_{\tau(i)}} \right)$$

and the total number of pairs with $i < j$ equals $1+2+\cdots+(g-1) = g(g-1)/2$. By (61) we obtain that

$$\begin{aligned}
\mathcal{D}_{p^*}(z) &= (\alpha_0^*)^{2g-2}(-1)^{\frac{g(g-1)}{2}} \prod_{i\neq j}\left(\frac{1}{w_{\tau(i)}} - \frac{1}{w_{\tau(j)}}\right) \\
&= (\alpha_0^*)^{2g-2}(-1)^{\frac{g(g-1)}{2}} \prod_{i\neq j}\left(\frac{1}{w_i} - \frac{1}{w_j}\right) \\
&= (\alpha_0^*)^{2g-2} \prod_{i<j}(w_i - w_j)^2 \times \prod_{i\neq j}\frac{1}{w_i w_j} \\
&= \left(\frac{\alpha_0^*}{N(w)}\right)^{2g-2} \prod_{i<j}(w_i - w_j)^2 = \left(\frac{\alpha_0^*}{N(w)}\right)^{2g-2} \mathcal{D}_p(z),
\end{aligned} \tag{62}$$

which extends to $\mathbb{C}$ by the uniqueness theorem. □

The following theorem describes invariants of algebraic points of $\mathcal{B}$ under Schur's algorithm.

Theorem 4.3. *Let f be an algebraic point of $\mathcal{B}$, w the corresponding algebraic function of order g, p the corresponding irreducible polynomial, $\{f_n\}_{n\geqslant 0}$ the Schur functions of f, $\{a_n\}_{n\geqslant 0}$ the Schur parameters. Then*

(a) *every Schur function f_n is an algebraic point of $\mathcal{B}$ of order g;*
(b) *$\mathcal{U}(f_n) = \mathcal{U}(f)$ for every n;*
(c) *the discriminant $\mathcal{D}(f_{n+1})$ divides $z^{(g-2)(g-1)}\mathcal{D}(f_n)$.*

Proof. In view of iterative character of Schur's algorithm it is sufficient to consider only its first iteration. Thus let

$$f(z) = \frac{zf_1(z) + a_0}{1 + \bar{a}_0 z f_1(z)}. \tag{63}$$

Substituting (63) in (38), we obtain

$$\alpha_0(zf_1(z)+a_0)^g + \alpha_1(zf_1+a_0)^{g-1}(1+\bar{a}_0 z f_1) + \cdots + \alpha_g(1+\bar{a}_0 z f_1)^g \equiv 0, \tag{64}$$

which shows that f_1 satisfies an algebraic equation of degree g. Notice that f_1 cannot satisfy an algebraic equation of degree smaller than g. Indeed, assuming the contrary, we can eliminate f_1 from this equation and from (63) and obtain an algebraic equation for f of degree smaller than g, which obviously contradicts to the irreducibility of p.

The direct computation yields the following formulae

$$\begin{aligned}
\beta_0 &= z^g\{\alpha_0(z) + \alpha_1(z)\bar{a}_0 + \cdots + \alpha_g(z)\bar{a}_0^g\}, \\
\beta_g &= \alpha_0(z)a_0^g + \alpha_1(z)a_0^{g-1} + \cdots + \alpha_g(z),
\end{aligned} \tag{65}$$

for the leading coefficient β_0 and for the free term β_g of the polynomial p_1 in f_1 in (64). Since by Lemma 2.4 both f and f_1 are the elements of C_A, (b) follows from the identity

$$1 - |f(z)|^2 = \frac{(1-|a_0|^2)(1-|f_1(z)|^2)}{|1+\bar{a}_0 z f_1(z)|^2}, \quad z \in \mathbb{T}.$$

Let $v_1, v_2, \ldots, v_g$ be the branches in $\mathbb{D}$ of the algebraic function, corresponding to the Schur function f_1. Let $\sigma^{(1)}, \sigma^{(2)}, \ldots, \sigma^{(g)}$ be the Galois automorphisms of $\mathbb{C}(z, w)$ over $\mathbb{C}(z)$ satisfying $w_k = \sigma^{(k)} f$, $k = 1, 2, \ldots, g$. Since any Galois automorphism keeps invariant elements of the coefficient field, we obtain from (63) that

$$w_k = \frac{z\sigma^{(k)} f_1 + a_0}{1 + \bar{a}_0 z\sigma^{(k)} f_1}, \quad k = 1, \ldots, g,$$

and we may assume that $v_k = \sigma^{(k)} f_1$. Hence

$$w_j - w_i = \frac{z(1 - |a_0|^2)(v_j - v_i)}{(1 + \bar{a}_0 z v_j)(1 + \bar{a}_0 z v_i)}.$$

Substituting this formula in (39), we obtain

$$\begin{aligned} D(f) &= {\alpha_0}^{2g-2} \prod_{i<j} (w_j - w_i)^2 \\ &= {\alpha_0}^{2g-2} \prod_{i<j} (v_j - v_i)^2 \frac{z(1 - |a_0|^2)}{(1 + \bar{a}_0 z v_i)^2} \frac{z(1 - |a_0|^2}{(1 + \bar{a}_0 z v_j)^2}. \end{aligned} \tag{66}$$

It is easy to see that each multiplier $z(1 - |a_0|^2)(1 + \bar{a}_0 z v_i)^{-2}$ enters the product in the right-hand side of (66) $g - 1$ times. Therefore,

$$D(f) = \prod_{i<j} (v_j - v_i)^2 \cdot {\alpha_0}^{2g-2} \cdot \left\{ \prod_{j=1}^{g} \frac{z}{1 - |a_0|^2} \left(\frac{1 - |a_0|^2}{1 + \bar{a}_0 z v_j} \right)^2 \right\}^{g-1}.$$

Using an obvious identity

$$1 - \bar{a}_0 w_j = \frac{1 - |a_0|^2}{1 + \bar{a}_0 z v_j},$$

and (65), we obtain

$$\begin{aligned} \mathcal{D}(f) &= \prod_{i<j} (v_j - v_i)^2 \cdot \left(\frac{z}{1 - |a_0|^2} \right)^{g(g-1)} \cdot \left\{ \alpha_0 \prod_{j=1}^{g} (1 - \bar{a}_0 w_j) \right\}^{2g-2} \\ &= \prod_{i<j} (v_j - v_i)^2 \cdot \left(\frac{z}{1 - |a_0|^2} \right)^{g(g-1)} \\ &\quad \times \{\alpha_0(z) + \alpha_1(z)\bar{a}_0 + \cdots + \alpha_g(z)\bar{a}_0^g\}^{2g-2} \\ &= z^{-g(g-1)} \cdot (1 - |a_0|^2)^{-g(g-1)} \cdot {\beta_0}^{2g-2} \cdot \prod_{i<j} (v_i - v_j)^2. \end{aligned}$$

Since $f(0) = a_0$ we have $\beta_g(0) = 0$. This and (64) imply that β_j/z are polynomials for $j = 0, \ldots, g$. Hence the greatest common divisor $\delta_f(z)$ of $\beta_0, \ldots, \beta_g$ is divisible by z and we obtain that

$$(1 - |a_0|^2)^{g(g-1)} \cdot z^{(g-2)(g-1)} \cdot \mathcal{D}(f) = \left\{ \frac{\delta_f(z)}{z} \right\}^{2g-2} \cdot \mathcal{D}(f_1), \tag{67}$$

which proves (c). $\square$

Corollary 4.4. *For large n the zeros of $\mathcal{D}(f_n)$ in $\mathbb{C} \setminus \{0\}$ taken with their multiplicities are stabilized.*

Proof. Iterating (67) we obtain

$$\prod_{j=0}^{n-1} (1-|a_j|^2)^{g(g-1)} z^{n(g-2)(g-1)} \mathcal{D}(f) = \prod_{j=0}^{n-1} \left\{ \frac{\delta_{f_j}}{z} \right\}^{2g-2} \mathcal{D}(f_n), \tag{68}$$

which shows that in a finite number of steps δ_{f_j} must be monomials $c_j z^{k_j}$ with $k_j \geqslant 1$. □

Corollary 4.5. *If $\mathcal{D}(f_n)$ are normalized to be monic polynomials, then there exists an integer N such that $|\mathcal{D}(f_n)|^2 = |\mathcal{D}(f_{n+1})|^2$ on $\mathbb{T}$ for every $n \geqslant N$.*

14. The following theorem provides some information on the behavior of Schur's parameters of algebraic point of $\mathcal{B}$.

Theorem 4.6. *Let f be an algebraic point of $\mathcal{B}$ and $\{a_n\}_{n \geqslant 0}$ Schur's parameters of f. Then either*

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty, \tag{69}$$

or there exist $\varepsilon > 0$ and a positive integer l such that

$$1 > \sup_{0 \leqslant j \leqslant l} |a_{n+j}| > \varepsilon, \quad n = 0, 1, \dots, \tag{70}$$

or f is a finite Blaschke product.

Proof. Let σ be a probability measure corresponding to the Schur function f. By Theorem 2.3 either σ is a Szegő measure, or $f \in \operatorname{Exp}(\mathcal{B})$. For any $f \in \mathcal{B}$ we have (see [12, Ch. IV, Exers.21, (d), 18])

$$\prod_{k=0}^{\infty} (1-|a_k|^2) = \exp\left\{ \int_{\mathbb{T}} \log(1-|f|^2)\, dm \right\}. \tag{71}$$

By (71) the first possibility implies (69). If $f \in \operatorname{Exp}(\mathcal{B})$ then either $\mathcal{U}(f)'$ is $\mathbb{T}$ and then f is a continuous inner function, i.e. a finite Blaschke product [21], or $|f| < 1$ on a non-empty open sub-arc J of $\mathbb{T}$ and $|f| = 1$ on a non-empty open arc I. In the later case $\operatorname{supp}(\sigma) \neq \mathbb{T}$, which by [19, Theorem 1.8] implies the existence of ε and l such that the right-hand side inequality of (70) holds. The left-hand side inequality holds by Rakhmanov lemma [29, Theorem 4.3.4] (see [18, Corollary 9.5] for a simple proof), which says that the condition

$$\overline{\lim_{n}}\, |a_n| = 1$$

implies that σ is a singular measure. □

It is useful to compare the above classification with a corresponding classification of regular continued fractions (10). Recall that a real number ξ is called a number of *constant type* (see

[22, Ch. II, §2]) if the partial denominators b_n are uniformly bounded. By Lagrange's theorem (Theorem B) every quadratic irrationality is a number of constant type. It is also clear that for every real ξ either

$$\overline{\lim_n}\, b_n = \infty \tag{72}$$

or ξ is a number of constant type, or ξ is rational. Notice, that the question of existence of algebraic ξ satisfying (72) as well as of the existence of algebraic numbers ξ of degree $n > 2$ in the set of numbers of constant type are well-known difficult problems of Diophantine Analysis.

To the contrary the same questions for Schur's algorithm have simple positive answers. By Theorem 4.6 condition (69) is an analogue of (72) for Schur's Algorithm. The linear fractional transform $z \to 1/(2-z)$ maps the unit circle $\mathbb{T}$ onto the horocycle of the diameter $2/3$ touching the circle $\mathbb{T}$ from inside at $z = 1$. Hence, $(2-z)^{-1/2}$ is an algebraic function corresponding to a Szegő measure. Similarly, just by taking algebraic roots one can obtain examples of algebraic exposed irrationalities of arbitrary high order, which correspond to regular continued fractions of constant type.

It is clear from the above arguments that exposed algebraic irrationalities correspond to irrational numbers of constant type, i.e. to real irrationalities which are badly approximable by rational numbers.

Theorem 3.1 suggests that a good candidate for the main object in the Euler–Lagrange Theory for Schur's Algorithm is the set

$$\operatorname{Exp}_q(\mathcal{B})$$

of all exposed points in $\mathcal{B}$, which are quadratic irrationalities. The fact that this choice is correct is supported by an observation that finite Blaschke products correspond to rational numbers. Being a quotient of two polynomials φ/φ^* they have finite Wall continued fractions and at the same time are obviously exposed points of $\mathcal{B}$. By Carathéodory's theorem [12, Ch. I, §2, Theorem 2.1] finite Blaschke products are dense in the $*$-weak topology of $\mathcal{B}$.

Another observation is that Wall continued fractions differ from regular continued fractions under the correspondence considered. Every real number with infinite regular continued fraction must be irrational. Schur's example $f(z) = 1/(2+z)$ (see (24)) demonstrates that a simple rational function may have an infinite Wall continued fraction. However, if we restrict our attention to the elements of $\operatorname{Exp}_q(\mathcal{B})$, then all these functions have infinite Wall fractions.

5. Exposed points in quadratic fields

15. A field $\mathbb{C}(z, w)$ of algebraic functions is called *quadratic* if w satisfies (38) with $g = 2$, i.e. there exist polynomials $a, b, c, \in \mathbb{C}[z]$ such that w satisfies the following irreducible (over $\mathbb{C}[z]$) quadratic equation

$$a(z)w^2(z) + b(z)w(z) + c(z) = 0. \tag{73}$$

Since (73) is assumed to be irreducible over $\mathbb{C}[z]$, the greatest common divisor of a, b, c in the commutative ring $\mathbb{C}[z]$ is a constant (see [8]). It follows from the definition that $a \not\equiv 0$ and $b^2 - 4ac$ is not the square of a polynomial.

The formula for the roots of quadratic equations shows that any extension field of $\mathbb{C}(z)$ of degree 2 is isomorphic to the splitting field $\mathbb{C}(z, \sqrt{\mathcal{D}})$ of the quadratic equation $X^2 - \mathcal{D} = 0$, $\mathcal{D}$

being a separable polynomial in z. The latter field can be identified with the set of all sums

$$w = x + y\sqrt{\mathcal{D}}; \quad x, y \in \mathbb{C}(z).$$

Given $w \in \mathbb{C}(z, \sqrt{\mathcal{D}})$ we denote by $w^{\#} \stackrel{\text{def}}{=} x - y\sqrt{\mathcal{D}}$ the algebraic conjugate element for w. The mapping $w \to w^{\#}$ is the only non-trivial automorphism of the Galois group $\mathrm{Gal}\left(\mathbb{C}(z, \sqrt{\mathcal{D}})/\mathbb{C}(z)\right)$. Clearly,

$$N(w) = w \cdot w^{\#} = x^2 - y^2\mathcal{D}; \quad Tr(w) = w + w^{\#} = 2x.$$

Using these formulas, we can write down explicitly the irreducible algebraic equation (over $\mathbb{C}(z)$) for a given element w of a quadratic field:

$$X^2 - Tr(w)X + N(w) = 0. \tag{74}$$

In this section we find necessary and sufficient conditions on the discriminant $\mathcal{D}$ of a quadratic field $\mathbb{C}(z, \sqrt{\mathcal{D}})$ in order that some of its elements could correspond to an algebraic exposed point of $\mathcal{B}$. A local version of this problem is also considered: find necessary and sufficient conditions on the coefficients of (73) in order that one of the roots be an exposed point of $\mathcal{B}$.

We sum up a number of necessary conditions in the following lemma.

Lemma 5.1. *Suppose that an algebraic function $w \in \mathbb{C}(z, \sqrt{\mathcal{D}})$, satisfying* (73), *corresponds to an algebraic exposed point $f \in \mathcal{B}$. Then*

(a) *$N(w)$ and $Tr(w)$ satisfy*

$$|N(w)| = 1; \quad Tr(w) = \overline{Tr(w)} \cdot N(w) \tag{75}$$

on $\mathbb{T}$;

(b) *$|a| = |c|$ on $\mathbb{T}$ and consequently the zeros of a and c coincide on $\mathbb{T}$*;

(c) *the polynomial b cannot vanish identically*;

(d) *a and b cannot have common zeros in* $\mathrm{Clos}(\mathbb{D})$.

Proof. Both identities (75) are immediate by Corollary 4.2. It follows from (73) and (74) that

$$Tr(w) = -\frac{b}{a}, \quad N(w) = \frac{c}{a},$$

which together with (75) imply that $|a| = |c|$ on $\mathbb{T}$.

To prove (c) assume to the contrary that $b \equiv 0$. Then we obtain by (74) that $f^2 = -N(w)$ in $\mathbb{D}$. It follows that $N(w)$ is a bounded holomorphic function in $\mathbb{D}$. Since $|N(w)| = 1$ on $\mathbb{T}$, we see that the rational function $N(w)$ is a finite Blaschke product, which implies that f is a finite Blaschke product too. This implies that w is a rational function, which is impossible, since we assume that Eq. (73) is irreducible.

It should be noticed at this place that our arguments are based on the assumption $f \in \mathcal{B}$. Similarly (d) holds under the same assumption. Indeed by Lemma 2.4 f extends to a continuous function on $\mathrm{Clos}\,\mathbb{D}$. Therefore, if $a(z_0) = b(z_0) = 0$ for z_0, $|z_0| \leqslant 1$, we obtain from (73) that $c(z_0) = 0$, which contradicts the assumption that (73) is irreducible over $\mathbb{C}[z]$. $\square$

Lemma 5.2. *Suppose that an algebraic function w, satisfying* (73), *corresponds to an algebraic exposed point f in $\mathcal{B}$. Let $L = \max(\deg a, \deg b, \deg c)$. Then there exists a unimodular constant λ such that*

$$(a)^*_L = \lambda c, \quad (b)^*_L = \lambda b, \quad (c)^*_L = \lambda a. \tag{76}$$

Proof. To simplify the notations we put for a time being $(p)^*_L = p^*$ for polynomials p with $\deg p \leqslant L$. By Theorem 4.1 we have $ba^{-1} = \bar{b}\bar{c}^{-1}$ on $\mathbb{T}$. It follows that

$$bc^* = ab^*, \quad b^*c = a^*b. \tag{77}$$

Suppose that the rational function b^*b^{-1} has a pole at $z_0 \in \mathbb{C}$. Then $b(z_0) = 0$. Let us rewrite (77) as follows

$$\frac{b^*}{b}a = c^*, \quad \frac{b^*}{b}c = a^*. \tag{78}$$

Since c^* and a^* are polynomials, we obtain that $a(z_0) = c(z_0) = 0$, which is impossible because $b(z_0) = 0$. It follows that b divides b^*.

Suppose now that bb^{*-1} has a pole at $z_0 \in \mathbb{C}$. Then $b^*(z_0) = 0$ and we obtain from (78) that $c^*(z_0) = a^*(z_0) = 0$. If $z_0 \neq 0$, then this implies that $1/\bar{z}_0$ is a common root for the triple (a, b, c), which is impossible. If $z_0 = 0$, then the equalities $a^*(0) = b^*(0) = c^*(0) = 0$ mean that $\max(\deg a, \deg b, \deg c) < L$, which contradicts our choice of L. It follows that b^* divides b.

Hence $b^* = \lambda b$, $\lambda \in \mathbb{C}$. We have $\lambda \in \mathbb{T}$, since $|b^*| = |b|$ on $\mathbb{T}$. Now other formulas (76) follow from (77) and $b^* = \lambda b$. □

Lemma 5.3. *Let $\mathbb{C}(z, \sqrt{\mathcal{D}})$ be a quadratic field defined by a separable polynomial $\mathcal{D}$. If $\mathbb{C}(z, \sqrt{\mathcal{D}})$ contains a quadratic irrationality w corresponding to an algebraic exposed point in $\mathcal{B}$, then the degree of $\mathcal{D}$ is even, all roots of $\mathcal{D}$ lie on $\mathbb{T}$ and* $\deg(\mathcal{D}) \leqslant 2L$.

Proof. Since there is a branch w of a quadratic irrationality in $\mathbb{C}(z, \sqrt{\mathcal{D}})$

$$w = x + y\sqrt{\mathcal{D}}; \quad x, y \in \mathbb{C}(z),$$

which is an exposed point for $\mathcal{B}$, $y \neq 0$, function w must be single-valued in $\mathbb{D}$. Hence $\mathcal{D}$ does not vanish in $\mathbb{D}$.

By Viéte's theorem we obtain from (73) and (74) that

$$x = -\frac{b}{2a}, \quad x^2 - y^2\mathcal{D} = \frac{c}{a},$$

which imply

$$b^2 - 4ac = 4a^2y^2\mathcal{D}. \tag{79}$$

Since $\mathcal{D}$ is separable and the left-hand side of (79) is a polynomial, the denominator of the rational function y must divide a. Hence

$$b^2 - 4ac = p^2\mathcal{D}, \quad p \in \mathbb{C}[z]. \tag{80}$$

By Lemma 5.2 we have on $\mathbb{T}$

$$b^2 - 4ac = \bar{\lambda}b(b)^*_L - 4\bar{\lambda}a(a)^*_L = \bar{\lambda}z^L(|b|^2 - 4|a|^2), \tag{81}$$

and therefore

$$(b^2 - 4ac)^*_{2L} = \lambda z^L(|b|^2 - 4|a|^2) = \lambda^2(b^2 - 4ac). \tag{82}$$

It follows that the zeros of $\mathcal{D}$ are symmetric with respect to $\mathbb{T}$. Since $\mathcal{D}$ does not vanish in $\mathbb{D}$, this implies that all roots of $\mathcal{D}$ are on $\mathbb{T}$.

Next, by (80) and (82) the polynomial

$$b^2 - 4ac, \quad \deg(b^2 - 4ac) \leqslant 2L - s$$

is λ^2-invariant under $p \to (p)^*_{2L}$. In particular, the coefficients $\{\gamma_j\}_{j=0}^{2L}$ of the polynomial $b^2 - 4ac$ satisfy

$$|\gamma_{L+j}| = |\gamma_{L-j}|, \quad j = 0, 1, \ldots, L.$$

Hence $\deg(b^2 - 4ac) = 2L - s$, where s is the order of the zero of $b^2 - 4ac = p^2\mathcal{D}$ (see (80)) at $z = 0$. Since $\mathcal{D}(0) \neq 0$, we obtain that the integer s is even and therefore $\deg(\mathcal{D})$ is even too. □

Lemma 5.4. *Let w be an algebraic function, satisfying an irreducible equation* (73) *with the discriminant $p^2\mathcal{D}$. If w corresponds to an exposed point f of $\mathcal{B}$, then*

(a) $\mathcal{U}(f) = \{z \in \mathbb{T};\ |Tr(w)| \leqslant 2\}$;
(b) *the zeros of b on $\mathbb{T}$ are located on $\mathcal{U}(f)$*;
(c) *there are no zeros of a on $\mathcal{U}(f)$*;
(d) *There is a continuous branch of $\sqrt{\mathcal{D}}$ along $\mathbb{T}$, bypassing the zeros of $\sqrt{\mathcal{D}}$ inside $\mathbb{D}$, such that*

$$\frac{p\sqrt{\mathcal{D}}}{b} \tag{83}$$

is positive on complementary arcs of the closed set $\mathcal{U}(f)$.

Proof. By Lemma 5.3 p is a polynomial and $\mathcal{D}$ is a separable polynomial of even degree with roots on $\mathbb{T}$. The formula for the roots of a quadratic equation shows that

$$2w_{1,2} = -\frac{b}{a}\left(1 \pm \frac{p\sqrt{\mathcal{D}}}{b}\right), \tag{84}$$

where p is a polynomial. By Lemma 5.2

$$\frac{p^2\mathcal{D}}{b^2} = 1 - \frac{4ac}{b^2} = 1 - \frac{4a^*a}{\lambda bb} = 1 - \frac{4a^*a}{b^*b} = 1 - \frac{4|a|^2}{|b|^2}$$

on $\mathbb{T}$. Hence $p\sqrt{\mathcal{D}}/b$ is pure imaginary on $\{z \in \mathbb{T};\ |b/a| \leqslant 2\}$, where by (84) both branches $w_{1,2}$ are unimodular. If $x = 2|a|/|b| < 1$, then

$$|w_{1,2}| = \frac{1 \pm \sqrt{1 - x^2}}{x} \quad \begin{cases} < 1 & \text{if } \pm = - \,; \\ > 1 & \text{if } \pm = + \,. \end{cases}$$

This proves (a). The statement (b) follows from (a) since $Tr(w) = 0$ if $b(z) = 0$. Similarly (c) follows from (a), since by Lemma 5.1 the polynomials b and a cannot have common zeros on $\mathbb{T}$.

By Lemma 2.4 one of the roots $w_1 = f$ belongs to the unit ball of the disc algebra. Therefore, there exists a continuous branch of $\sqrt{D}$ such that

$$f = w_1 = -\frac{b}{2a}\left(1 - \frac{p\sqrt{\mathcal{D}}}{b}\right)$$

and $p\sqrt{D}/b > 0$ on $\mathbb{T} \setminus \mathcal{U}(f)$. This implies (d). □

16. Suppose that there is an exposed irrationality satisfying (73) with

$$b^2 - 4ac = \mathcal{D},$$

where the discriminant $\mathcal{D}$ of (73) is a separable polynomial. Then this irrationality is an element of the quadratic function field $\mathbb{C}(z, \mathcal{D})$. By Lemma 5.3 $\deg(\mathcal{D}) = 2L$ and the roots of $\mathcal{D}$ are located on $\mathbb{T}$. In particular, $\mathcal{D}(0) \neq 0$. Next, by (82) we have $\mathcal{D}^* = \lambda^2\mathcal{D}$ where λ is a unimodular constant satisfying (76). It follows that

$$T_{\mathcal{D}}(z) = \lambda\bar{z}^L\mathcal{D}(z), \quad z \in \mathbb{T}, \tag{85}$$

is a real trigonometric polynomial on $\mathbb{T}$ of degree L. All $2L$ zeros of $T_{\mathcal{D}}$ are identical with zeros of $\mathcal{D}$ and therefore are simple. By (81) we obtain a useful formula

$$T_{\mathcal{D}}(z) = |b(z)|^2 - 4|a(z)|^2, \quad z \in \mathbb{T}. \tag{86}$$

By Lemma 5.4(a)

$$\mathcal{U}(f) = \{z \in \mathbb{T} : T_{\mathcal{D}}(z) \leqslant 0\}.$$

Since $T_{\mathcal{D}}$ has $2L$ simple zeros, $\mathcal{U}(f)$ is the union of L closed arcs $\{\gamma_j\}_{j=1}^L$, $\gamma_j = [t_j^-, t_j^+]$, $j = 1, \ldots, L$. Here $\{t_j^-\}_{j=1}^L$ and $\{t_j^+\}_{j=1}^L$ are two interlaced sequences of the zeros of $\mathcal{D}$ numbered counterclockwise. The complement $\mathcal{E}(f) \overset{\text{def}}{=} \mathbb{T} \setminus \mathcal{U}(f)$ can be presented as

$$\mathcal{E}(f) = \bigcup_{j=1}^{L} \omega_j, \tag{87}$$

where $\omega_j = (t_j^+, t_{j+1}^-)$, $j = 1, \ldots, L$, $t_{L+1}^- = t_1^-$.

By Lemma 5.1(d) polynomials a and b cannot both vanish on $\mathbb{T}$. It follows that the roots of b on $\mathbb{T}$ must lie in the open arcs (t_j^-, t_j^+), $j = 1, \ldots, L$. Let us fix a closed arc $\gamma_j = [t_j^-, t_j^+]$ and two arbitrary points ξ_{j-1} and ξ_j in the open arcs ω_{j-1} and ω_j adjacent to γ_j.

Lemma 5.5. *Let n_j be the number of zeros (counting multiplicities) of b in (t_j^-, t_j^+). Let Γ be any path in $\mathbb{D}$ starting at ξ_{j-1} and terminating at ξ. We assume that the support of Γ is sufficiently close to $\mathbb{T}$. Then*

$$\frac{1}{2\pi}\Delta_{\Gamma}\arg\sqrt{\frac{\mathcal{D}}{b^2}} = \frac{n_j}{2} - \frac{1}{2}. \tag{88}$$

Proof. We apply the arguments similar to those used in [26, Lemma 3.1]. Let

$$G_{\rho,j} = \{z \in \mathbb{D} : \rho < |z| < 1, \arg\xi_{j-1} < \arg z < \arg\xi_j\}.$$

If ρ is sufficiently close to 1, then $\mathcal{D}b^{-2}$ has no poles in a simple connected domain $G_{\rho,j}$. By the Monodromy theorem [8, Ch. III, §1, Theorem 1.2, 24, Theorem 8.5, p. 269_3] we may replace the path Γ with an auxiliary path from ξ_{j-1} to ξ_j which moves along $\mathbb{T}$ and bypasses the zeros and poles of $\mathcal{D}b^{-2}$ along small semi-circles in $\mathbb{D}$ centered at these singularities. By (85) and (86) we have on $[\xi_{j-1}, \xi_j]$:

$$\frac{\mathcal{D}(z)}{b^2(z)} = \frac{\bar{\lambda} z^L T_{\mathcal{D}}(z)}{\bar{\lambda} b^* b} = \frac{|b|^2 - 4|a|^2}{|b|^2},$$

which implies that $\mathcal{D}b^{-2}$ is real on $[\xi_{j-1}, \xi_j]$ with positive values at the ends. Two zeros t_j^- and t_j^+ contribute $-\pi - \pi = -2\pi$ in the increment of the argument of $\mathcal{D}b^{-2}$ along Γ. On the other hand, the poles of $\mathcal{D}b^{-2}$ contribute $2\pi n_j$ in the increment considered. Hence the total increment obtained is given by (88). □

Corollary 5.6. *Let w be a quadratic irrationality corresponding to an exposed point of $\mathcal{B}$ and satisfying* (73) *with separable discriminant $\mathcal{D}$. Then b is a separable polynomial of degree $L = \deg(\mathcal{D})/2$ such that every arc (t_j^-, t_j^+), $j = 1, \ldots, L$, contains exactly one root of b.*

Proof. Both branches

$$\pm\frac{\sqrt{\mathcal{D}}}{b} = \pm\sqrt{\frac{\mathcal{D}}{b^2}}$$

are single-valued analytic functions in a simple connected domain $G_{\rho,j}$. By Lemma 5.5 the increment of the argument of each branch along a continuous path Γ in $G_{\rho,j}$ from ξ_{j-1} to ξ_j is $\pi(n_j - 1)$. By Lemma 5.4 these signs must be equal. By Lemma 5.5 the number n_j of the zeros of b in (t_j^-, t_j^+) must be odd. Taking into account that by Lemma 5.2 $\deg(b) \leqslant L$, we see that the total number of zeros of b cannot exceed L. On the other hand, the total number of arcs (t_j^-, t^+) is L. Hence $n_j = 1$, $j = 1, \ldots, L$. □

Theorem 5.7. *A quadratic field $\mathbb{C}(z, \sqrt{\mathcal{D}})$ contains an element corresponding to an exposed point of $\mathcal{B}$ if and only if $\mathcal{D}$ is a separable polynomial of even degree with roots on $\mathbb{T}$.*

Proof. The necessity follows by Lemma 3.3. Suppose now that $\deg \mathcal{D} = 2L$, that all roots $\{t_j^{\pm}\}_{j=1}^L$ of $\mathcal{D}$ are simple and lie on $\mathbb{T}$. It follows that $\mathcal{D} \in {}^*\mathbb{C}[z]$, see (35), and therefore

$$\mathcal{D}^* = \lambda(\mathcal{D})\mathcal{D}.$$

The equation $\lambda^2 = \lambda(\mathcal{D})$ has two roots $\lambda_{1,2} = \pm\sqrt{\lambda(\mathcal{D})}$. For λ_1 the real trigonometric polynomial

$$T_{\mathcal{D}}(z) = \lambda_1 \bar{z}^L \mathcal{D}(z), \quad z \in \mathbb{T},$$

is negative on the system of arcs (t_j^-, t_j^+), $j = 1, \ldots, L$, whereas for $\lambda_2 = -\lambda_1$ the polynomial $T_{\mathcal{D}} = \lambda_2 \bar{z}^L \mathcal{D}$ is negative on the complementary arcs. Our arguments do not depend on this alternative. Therefore, we assume that $\lambda_1 = \lambda$, $\lambda^2 = \lambda(\mathcal{D})$.

Let us pick L points $\xi_1, \ldots, \xi_L$ on $\mathbb{T}$, satisfying

$$\xi_j \in (t_j^-, t_j^+), \quad j = 1, \ldots, L,$$

and define an auxiliary polynomial of degree L by

$$\Delta_L(z) = rt \prod_{j=1}^{L} (z - \xi_j), \tag{89}$$

where $r > 0$ and $t \in \mathbb{T}$ are some constants to be fixed later. By (35) we have

$$\lambda(\Delta_L) = \bar{t}^2 \prod_{j=1}^{L} (-\bar{\xi}_j) = \lambda, \quad \lambda^2 = \lambda(\mathcal{D}) \tag{90}$$

for two values of t. Since $T_{\mathcal{D}}$ is negative on the arcs (t_j^-, t_j^+), $j = 1, \ldots, L$, the difference

$$|\Delta_L|^2 - T_{\mathcal{D}} \tag{91}$$

is strictly positive on these arcs. On the complementary arcs $[t_j^+, t_{j+1}^-]$, $j = 1, \ldots, L$, $t_{L+1}^- = t_1^-$, the trigonometric polynomial $-T_{\mathcal{D}}$ is not positive, whereas the trigonometric polynomial $|\Delta_L|^2$ is strictly positive. Therefore, increasing r, we can force the difference (91) to be positive on $\mathbb{T}$. By Fejér's theorem [31, Theorem 1.2.2] there exists a polynomial a in z of degree L, such that a does not vanish in the closed unit disk and satisfies $|\Delta_L|^2 - T_{\mathcal{D}} = 4|a|^2$ on $\mathbb{T}$. Let us take the value of t in (89) to satisfy (90) and define $b = \Delta_L$. By (90) we have $\lambda^2 = \lambda(\mathcal{D})$. Next, we put $c = \overline{\lambda(b)}a^*$. Then

$$T_{\mathcal{D}} = |b|^2 - 4|a|^2 = \bar{z}^L\{bb^* - 4aa^*\} = \lambda(b)\bar{z}^L\{b^2 - 4ac\}.$$

Since on the other hand $T_{\mathcal{D}} = \lambda(b)\bar{z}^L\mathcal{D}(z)$, we obtain that

$$b^2 - 4ac = \mathcal{D}$$

holds on the complex plane. It follows that $\mathcal{D}$ is the discriminant of (73) corresponding to the triple (a, b, c). In particular, (73) is irreducible. The zeros of b are located on $\mathbb{T}$, whereas $|a| = |c|$ is strictly positive on $\mathbb{T}$. Hence, the greatest common divisor of (a, b, c) is a constant. The roots of (73) in $\mathbb{D}$ are given by

$$w_{1,2}(z) = \frac{-b(z) \pm \sqrt{\mathcal{D}}}{2a(z)}.$$

Since a does not vanish in $\operatorname{Clos}\mathbb{D}$ by the construction, both roots w_1 and w_2 belong to the disc algebra C_A. By the maximum modulus theorem one of the functions w_1, w_2 is in $\mathcal{B}$ if

$$|-b \pm \sqrt{\mathcal{D}}|^2 \leqslant 4|a|^2 \tag{92}$$

on $\mathbb{T}$ for one of two branches $\pm\sqrt{\mathcal{D}}$. Simple algebra shows that (92) is equivalent to

$$|b|^2 - 4|a|^2 + |\mathcal{D}| \leqslant \pm 2|b|^2 \Re\sqrt{\frac{\mathcal{D}}{b^2}}. \tag{93}$$

Suppose first that $z \in [t_j^-, t_j^+]$, $j = 1, \ldots, L$. Then $T_{\mathcal{D}} = |b|^2 - 4|a|^2 \leqslant 0$, which implies that the left-hand side of (93) is zero, since $|T_{\mathcal{D}}| = |\mathcal{D}|$. Next, we have on $\mathbb{T}$

$$\frac{\mathcal{D}}{b^2} = 1 - 4\frac{ac}{bb} = 1 - 4\frac{\overline{\lambda(b)}aa^*}{\overline{\lambda(b)}bb^*} = 1 - 4\left|\frac{a}{b}\right|^2 = \frac{T_{\mathcal{D}}}{|b|^2}. \tag{94}$$

Since $T_{\mathcal{D}} \leqslant 0$ on $[t_j^-, t_j^+]$, we obtain that the right-hand side of (93) is zero too. It follows that both branches are unimodular on $[t_j^-, t_j^+]$, $j = 1, \dots, L$.

Suppose now that $z \in \omega_j = (t_j^+, t_{j+1}^-)$. Then $T_{\mathcal{D}} > 0$ and the left-hand side of (93) equals $2|T_{\mathcal{D}}|$, whereas the modulus of the right-hand side of (93) equals $2|b|\sqrt{|T_{\mathcal{D}}|}$ by (94). Since obviously

$$\sqrt{|T_{\mathcal{D}}|} = \sqrt{|b|^2 - 4|a|^2} \leqslant |b|$$

on ω_j, we obtain that (93) holds for the branch of the root $\pm\sqrt{\mathcal{D}b^{-2}}$, which is positive on every ω_j. By (89) b has L simple zeros on $\mathbb{T}$ and every arc (t_j^-, t_j^+) contains exactly one zero. Applying Lemma 5.5, we conclude that such a branch exists, since $n_j = 1$, $j = 1, \dots, L$. □

It is easy to see that in fact we proved more than it has been stated.

Corollary 5.8. *Let $\mathcal{D}$ be a separable polynomial with roots on $\mathbb{T}$, $\deg(\mathcal{D}) = 2L$. Then there are polynomials a, b, c, of degree L, such that every arc (t_j^-, t_j^+), $j = 1, \dots, L$ contains exactly one simple zero of b, a is invertible in the disc algebra C_A, $c = \overline{\lambda(b)}a^*$, $D = b^2 - 4ac$, and one of the roots of* (73) *with the coefficients* (a, b, c) *corresponds to an exposed point of $\mathcal{B}$.*

Corollary 5.9. *Let $\mathcal{D}$ be a separable polynomial with roots on $\mathbb{T}$, $\deg(\mathcal{D}) = 2L$. Then a separable polynomial b of degree L can be the middle coefficient of the irreducible equation for an exposed quadratic irrationality with the discriminant $\mathcal{D}$ if and only if every of L arcs constituting the set $\{t \in \mathbb{T} : T_{\mathcal{D}}(t) < 0\}$ contains exactly one root of b and $|b|^2 \geqslant T_{\mathcal{D}}$ on $\mathbb{T}$.*

Corollary 5.10. *For any family of disjoint closed arcs $\{\gamma_j\}_{j=1}^L$ there is an exposed quadratic irrationality f in $\mathcal{B}$ such that*

$$\mathcal{U}(f) = \bigcup_{j=1}^{L} \gamma_j.$$

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